Composition operators on the Wiener-Dirichlet algebra

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Abstract. *We study the composition operators on an algebra of Dirichlet series, the analogue of the Wiener algebra of absolutely convergent Taylor series, which we call the Wiener-Dirichlet algebra. The central issue is to understand the connection between the properties of the operator and of its symbol, with special emphasis on the compact, automorphic, or isometric character of this operator. We are led to the intermediate study of algebras of functions of several, or countably many, complex variables.*

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1 Introduction

Let $A^+ = A^+(\mathbb{T})$ be the Wiener algebra of absolutely convergent Taylor series in one variable : $f \in A^+$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{with } \|f\|_{A^+} = \sum_{n=0}^{\infty} |a_n| < +\infty.$$

It is well-known that A^+ is a commutative, unital Banach algebra with spectrum $\overline{\mathbb{D}}$, the closed unit disk. If $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic, the composition operator C_ϕ with symbol ϕ is formally defined by $C_\phi(f) = f \circ \phi$.

Newman [20] studied those symbols ϕ generating bounded composition operators $C_\phi \colon A^+ \to A^+$, and proved in particular the following:

(a) C_ϕ maps A^+ into itself if and only if $\phi \in A^+$ and $\|\phi^n\|_{A^+} = O\,(1)$ as $n \to \infty$ (*e.g.* $\phi(z) = 5^{-1/2}(1 + z - z^2)$): this happens if and only if all maximum points θ_0 of $|\phi(\mathrm{e}^{i\theta})|$ are "*ordinary points*", *i.e.* if and only if we have, as $t \to 0$:

$$\log \phi(\mathrm{e}^{i(\theta_0+t)}) = \alpha_0 + \alpha_1 t + \alpha_k t^k + \cdots,$$

where $k > 1$ and $\alpha_k \neq 0$ is not purely imaginary;

(b) if moreover $|\phi(e^{it})| = 1$, one must have $\phi(z) = az^d$, with $|a| = 1$ and $d \in \mathbb{N}$;

(c) $C_\phi \colon A^+ \to A^+$ is an automorphism if and only if $\phi(z) = az$, with $|a| = 1$.

Harzallah (see [14]) also proved that:

(d) $C_\phi \colon A^+ \to A^+$ is an isometry if and only if $\phi(z) = az^d$, with $|a| = 1$ and $d \in \mathbb{N}$.

The aim of this paper is to perform a similar study for the "*Wiener-Dirichlet*" algebra $\mathcal{A}^+$ of absolutely convergent Dirichlet series: $f \in \mathcal{A}^+$ if and only if

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{with } \|f\|_{\mathcal{A}^+} = \sum_{n=1}^{\infty} |a_n| < +\infty.$$

$\mathcal{A}^+$ is a commutative, unital Banach algebra, with the following multiplication (quite different from the one for Taylor series):

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}, \text{ with } c_n = \sum_{ij=n} a_i b_j.$$

$\mathcal{A}^+$ can also be interpreted as a space of analytic functions on $\mathbb{C}_0$ (where in general we denote by $\mathbb{C}_\theta$ the vertical half-plane $\mathcal{R}e\, s > \theta$). The study of function spaces formed by Dirichlet series has gained some recent interest (see the papers of Hedenmalm-Lindqvist-Seip [10], Gordon-Hedenmalm [8], Bayart [1], [2], Finet-Queffélec-Volberg [7], Finet-Queffélec [6], Finet-Li-Queffélec [5], Mc Carthy [18]). Now, a method due to Bohr (see for example [21]) identifies the algebra $\mathcal{A}^+$ with the algebra $A^+(\mathbb{T}^\infty)$ formed by the absolutely convergent Taylor series in countably many variables (this point of view, which allows to identify the spectrum of $\mathcal{A}^+$ as $\overline{\mathbb{D}}^\infty$, the spectrum of $A^+(\mathbb{T}^\infty)$, has been used by Hewitt and Williamson [11], among others, to prove the following Wiener type tauberian Theorem : "*If $f \in \mathcal{A}^+$ and $|f(s)| \geqslant \delta > 0$ for $s \in \mathbb{C}_0$, then* $1/f \in \mathcal{A}^+$").

Let us recall the way this identification is carried out. Let $(p_j)_{j\geqslant 1}$ be the increasing sequence of prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$). If

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\infty)}} a_\alpha z^\alpha \quad \text{with } \|f\|_{A^+(\mathbb{T}^\infty)} = \sum_{\alpha} |a_\alpha| < +\infty,$$

where, as usual, we set $\alpha = (\alpha_1, \ldots, \alpha_r, 0, 0, \ldots)$ and $z^\alpha = z_1^{\alpha_1} \ldots z_r^{\alpha_r}$ for $z = (z_j)_{j\geqslant 1}$, then $\Delta \colon \mathcal{A}^+ \to A^+(\mathbb{T}^\infty)$ is defined by:

$$\Delta\left(\sum_{n=1}^{\infty} a_n n^{-s}\right) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \ldots z_r^{\alpha_r},$$

if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is the decomposition of n in prime factors. Δ is an isometric isomorphism. Moreover, we shall need two more facts about Δ. For $s \in \mathbb{C}_0$, we set $z^{[s]} = (p_j^{-s})_j$. We then have:

$$\begin{aligned} \Delta f(z^{[s]}) &= f(s), \text{ for any } f \in \mathcal{A}^+ \text{ and any } s \in \mathbb{C}_0 & (1) \\ \|\Delta f\|_\infty &= \|f\|_\infty \text{ for each } f \in \mathcal{A}^+, & (2) \end{aligned}$$

where we set $\|f\|_\infty = \sup_{s \in \mathbb{C}_0} |f(s)|$ and $\|\Delta f\|_\infty = \sup_{z \in \mathbf{B}} |\Delta f(z)|$, with $\mathbf{B} = \{z = (z_j)_{j \geqslant 1} \in \mathbb{D}^\infty \,;\, z_j \underset{j \to +\infty}{\longrightarrow} 0\}$. Indeed, if $f(s) = \sum_1^\infty a_n n^{-s}$, we have:

$$\Delta f(z^{[s]}) = \sum_{n=1}^{\infty} a_n (p_1^{-s})^{\alpha_1} \dots (p_r^{-s})^{\alpha_r} = \sum_{n=1}^{\infty} a_n (p_1^{\alpha_1} \dots p_r^{\alpha_r})^{-s} = f(s).$$

On the other hand, let $z = (z_j)_{j \geqslant 1} \in \mathbf{B}$. Fix an integer N, let $k = \pi(N)$ be the number of primes not exceeding N, and $S_N(z) = \sum_{n=1}^N a_n z_1^{\alpha_1} \dots z_k^{\alpha_k}$, with $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Pick $\sigma > 0$ such that $|z_j| \leqslant p_j^{-\sigma}$, $1 \leqslant j \leqslant k$. Due to the rational independence of $\log p_1, \dots, \log p_k$ and to the Kronecker Approximation Theorem ([12], Corollary 4, page 23), the points $(p_j^{-it})_{1 \leqslant j \leqslant k}$, $t \in \mathbb{R}$, are dense in the torus $\mathbb{T}^k$, so that the maximum modulus principle for the polydisk $\mathbb{D}^k$ gives:

$$\begin{aligned} |S_N(z)| &\leqslant \sup_{|w_j| = p_j^{-\sigma}} \Big| \sum_{n=1}^{N} a_n w_1^{\alpha_1} \dots w_k^{\alpha_k} \Big| = \sup_{\mathcal{R}e\, s = \sigma} \Big| \sum_{n=1}^{N} a_n (p_1^{-s})^{\alpha_1} \dots (p_k^{-s})^{\alpha_k} \Big| \\ &= \sup_{\mathcal{R}e\, s = \sigma} \Big| \sum_{n=1}^{N} a_n n^{-s} \Big| \leqslant \Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_\infty . \end{aligned}$$

Hence $\|S_N\|_\infty \leqslant \| \sum_1^N a_n n^{-s} \|_\infty$. Letting N tend to infinity gives $\|\Delta f\|_\infty \leqslant \|f\|_\infty$, which proves (2), since we trivially have $\|\Delta f\|_\infty \geqslant \|f\|_\infty$.

In this paper, we use the identification proposed above to obtain results similar to (a), (b), (c) and (d) for $\mathcal{A}^+$. This leads to an intermediate study of composition operators on the algebras $A^+(\mathbb{T}^\infty)$ and $A^+(\mathbb{T}^k)$ (the k-dimensional analog of $A^+(\mathbb{T}^\infty)$). Accordingly, the paper is organized as follows:
In Section 2, we give necessary as well as sufficient conditions for boundedness and compactness of $C_\phi : \mathcal{A}^+ \to \mathcal{A}^+$, and study in detail some specific examples. In Section 3, we study the automorphisms of the algebras $A^+(\mathbb{T}^k)$, $A^+(\mathbb{T}^\infty)$, $\mathcal{A}^+$. In Section 4, we study the isometries of those algebras, and we point out some specific differences between the finite and infinite-dimensional cases. Section 5 is devoted to some concluding remarks and questions.

A word on the definitions and notations: we will say that integers $2 \leqslant q_1 < q_2 < \dots$ are multiplicatively independent if their logarithms are rationally independent in the real numbers; equivalently, if any integer $n \geqslant 2$ can be expressed as $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, $\alpha_j \in \mathbb{N}_0$, in at most one way (e.g. $q_1 = 2$, $q_2 = 6$, $q_3 = 30$). We shall denote by $\mathcal{D}$ the space of functions $\varphi \colon \mathbb{C}_0 \to \mathbb{C}$ which are analytic, and moreover representable as a convergent Dirichlet series $\sum_1^\infty c_n n^{-s}$ for $\mathcal{R}e\, s$ large enough. $\mathcal{D}$ is also called the space of convergent Dirichlet series. For example, if $\psi(s) = (1 - 2^{1-s})\zeta(s)$, and $\varphi(s) = \psi(s - a)$, φ is entire, and representable as $\sum_1^\infty (-1)^{n-1} n^a n^{-s}$ for $\mathcal{R}e\, s > a$. $\mathbb{T}$ denotes the unit circle, and plays no role in the definition of $A^+(\mathbb{T}^k)$ and $A^+(\mathbb{T}^\infty)$, although $\mathbb{T}^k$ (resp. $\mathbb{T}^\infty$) might be viewed as the Šhilov boundary of $A^+(\mathbb{T}^k)$ (resp. $A^+(\mathbb{T}^\infty)$). As usual, we set $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. Recall that $\mathbb{C}_\theta$ is the vertical half-plane $\mathcal{R}e\, s > \theta$.

2 Boundedness and Compactness of Composition Operators $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$

2.1 General results

We begin by sharpening Newman's result ((a) of the Introduction), under the form of the following (where it is assumed that ϕ is non-constant):

Proposition 1 *The composition operator $C_\phi\colon A^+ \to A^+$ is compact if and only if $\|\phi\|_\infty = \sup_{z\in\mathbb{D}} |\phi(z)| < 1$.*

Proof. As will be apparent from the proof of the next Proposition, $C_\phi\colon A^+ \to A^+$ is compact if and only if $\|\phi^n\|_{A^+} \to 0$ as $n \to \infty$. On the other hand, by the spectral radius formula, we have $\|\phi\|_\infty = \lim_{n\to\infty} \|\phi^n\|_{A^+}^{1/n} = \inf_{n\geqslant 1} \|\phi^n\|_{A^+}^{1/n}$. That finishes the proof.

Alternatively, we could have applied to $f_n(z) = z^n$ a general criterion of Shapiro [24]: "*C_ϕ is compact if and only if $\|C_\phi(f_n)\|_{A^+} \to 0$ for each sequence $(f_n)_n$ in A^+ which is bounded in norm and converges uniformly to zero on compact subsets of $\mathbb{D}$*". □

We now turn to the study of composition operators $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ associated with an analytic function $\phi\colon \mathbb{C}_0 \to \mathbb{C}_0$.

We first recall the following:

Theorem 2 *([8, Theorem 4]). Let $\phi\colon \mathbb{C}_0 \to \mathbb{C}$ be an analytic function such that $k^{-\phi} \in \mathcal{D}$ for $k = 1, 2, \ldots$. Then we have necessarily:*

$$\phi(s) = c_0 s + \varphi(s), \text{ with } c_0 \in \mathbb{N}_0 \text{ and } \varphi \in \mathcal{D}. \tag{3}$$

We will therefore restrict ourselves, in the sequel, to symbols ϕ of the form given by (3). To avoid trivialities, we will also assume once and for all that ϕ is non-constant.

Theorem 3 *Let $\phi\colon \mathbb{C}_0 \to \mathbb{C}$ be an analytic function of the form* (3)*. Then :*

(a) *(i) if C_ϕ maps $\mathcal{A}^+$ into itself then $n^{-\phi} \in \mathcal{A}^+$ and $\|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, $n = 1, 2, \ldots$, for a positive constant C independent of n;*

(ii) conversely, if $(n^{-\phi})_{n=1}^\infty$ is a bounded sequence in $\mathcal{A}^+$, then ϕ maps $\mathbb{C}_0$ into $\mathbb{C}_0$ and C_ϕ is a bounded composition operator on $\mathcal{A}^+$.

(b) *(i) $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact if and only if $\|n^{-\phi}\|_{\mathcal{A}^+} \underset{n\to\infty}{\longrightarrow} 0$. Then $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ for some $\delta > 0$.*

(ii) Assume that $\phi(s) = c_0 s + \sum_1^\infty c_n n^{-s}$, with $\sum_1^\infty |c_n| < +\infty$. Then C_ϕ is compact if and only if $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ for some $\delta > 0$.

Proof. (a) (i) Suppose that C_ϕ maps $\mathcal{A}^+$ into itself. C_ϕ is an algebra homomorphism and $\mathcal{A}^+$ is semi-simple, therefore (see [22, p. 263]) C_ϕ is continuous. Thus:

$$\|n^{-\phi}\|_{\mathcal{A}^+} = \|C_\phi(n^{-s})\|_{\mathcal{A}^+} \leqslant \|C_\phi\|\, \|n^{-s}\|_{\mathcal{A}^+} = \|C_\phi\| =: C.$$

(ii) Conversely, suppose that $n^{-\phi} \in \mathcal{A}^+$ and $\|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, $n = 1, 2, \ldots$. We first see that, for $s \in \mathbb{C}_0$, we have: $n^{-\mathcal{R}e\,\phi(s)} = |n^{-\phi(s)}| \leqslant \|n^{-\phi}\|_\infty \leqslant \|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, whence $\mathcal{R}e\,\phi(s) \geqslant -\frac{\log C}{\log n}\cdot$ Letting n tend to infinity gives $\mathcal{R}e\,\phi(s) \geqslant 0$, and the open mapping theorem gives $\mathcal{R}e\,\phi(s) > 0$, since ϕ is not constant. If now $f(s) = \sum_1^\infty a_n n^{-s} \in \mathcal{A}^+$, the series $\sum_1^\infty a_n n^{-\phi(s)}$ is absolutely convergent in $\mathcal{A}^+$, so that $f \circ \phi \in \mathcal{A}^+$, with $\|f \circ \phi\|_{\mathcal{A}^+} \leqslant \sum_1^\infty |a_n| \|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C \sum_1^\infty |a_n| = C\|f\|_{\mathcal{A}^+}$.

(b) (i) Suppose that $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact. Let $f \in \mathcal{A}^+$ be a cluster point of $n^{-\phi(s)} = C_\phi(n^{-s})$, and let $(n_k)_k$ be a sequence of integers such that $\|n_k^{-\phi} - f\|_{\mathcal{A}^+} \to 0$. For fixed $s \in \mathbb{C}_0$, we have $|n_k^{-\phi(s)} - f(s)| \leqslant \|n_k^{-\phi} - f\|_{\mathcal{A}^+}$. But $n_k^{-\phi(s)} \to 0$ (since $\mathcal{R}e\,\phi(s) > 0$, by part (a)), so that $f(s) = 0$. Hence $f = 0$. This implies $\|n^{-\phi}\|_{\mathcal{A}^+} \to 0$.

Now, since $\|n^{-\phi}\|_\infty \leqslant \|n^{-\phi}\|_{\mathcal{A}^+}$, we get $n^{-\inf \mathcal{R}e\,\phi(s)} = \|n^{-\phi}\|_\infty \to 0$, and so $\inf\limits_{s\in\mathbb{C}_0} \mathcal{R}e\,\phi(s) > 0$.

Conversely, suppose that $\epsilon_n = \|n^{-\phi}\|_{\mathcal{A}^+} \to 0$ and set $\delta_n = \sup_{k>n} \epsilon_k$. Let $T_n\colon \mathcal{A}^+ \to \mathcal{A}^+$ be the finite-rank operator defined by $(T_n f)(s) = \sum_{k=1}^n a_k k^{-\phi(s)}$ if $f(s) = \sum_{k=1}^\infty a_k k^{-s}$. We have:

$$\|C_\phi f - T_n f\|_{\mathcal{A}^+} \leqslant \sum_{k>n} |a_k| \|k^{-\phi}\|_{\mathcal{A}^+} \leqslant \delta_n \sum_{k>n} |a_k| \leqslant \delta_n \|f\|_{\mathcal{A}^+},$$

showing that $\|C_\phi - T_n\| \leqslant \delta_n$, and therefore that C_ϕ is compact.

(ii) For any $\upsilon \in \mathcal{A}^+$, and for any real number $r \geqslant 1$, we have:

$$\|r^{-\upsilon}\|_{\mathcal{A}^+} \leqslant r^{\|\upsilon\|_{\mathcal{A}^+}}\,. \tag{4}$$

Indeed:

$$r^{-\upsilon} = \exp(-\upsilon \log r) = \sum_{k=0}^\infty \frac{(-\log r)^k}{k!}\, \upsilon^k \in \mathcal{A}^+$$

since υ belongs to the algebra $\mathcal{A}^+$. Moreover:

$$\|r^{-\upsilon}\|_{\mathcal{A}^+} \leqslant \sum_{k=0}^{\infty} \frac{(\log r)^k}{k!} \|\upsilon\|_{\mathcal{A}^+}^k = r^{\|\upsilon\|_{\mathcal{A}^+}}$$

(we may remark that when $\upsilon(s) = c_j j^{-s}$ is a monomial, we have equality; in particular: $\|n^{-c_j j^{-s}}\|_{\mathcal{A}^+} = n^{|c_j|}$ for every positive integer n).

We shall use the following:

Proposition 4 *(see [8]) Let θ and τ be real numbers and suppose that ϕ maps $\mathbb{C}_\theta$ into $\mathbb{C}_\tau$. Then, if $\phi(s) = c_0 s + \varphi(s)$, and φ is not constant, φ maps $\mathbb{C}_\theta$ into $\mathbb{C}_{\tau - c_0\theta}$.*

Now, assume that φ is non-constant (since otherwise the result is trivial), and that $\varepsilon = \inf_{s\in\mathbb{C}_0} \mathcal{R}e\, \phi(s) > 0$. By Proposition 4, φ maps $\mathbb{C}_0$ into $\mathbb{C}_\varepsilon$. The spectral radius formula and Bohr's theory (as seen in the Introduction) give, with $\psi = 2^{-\varphi}$:

$$\lim_{j\to+\infty} \|\psi^j\|_{\mathcal{A}^+}^{1/j} = \sup_{h\in \mathrm{sp}\, \mathcal{A}^+} |h(\psi)| = \sup_{s\in\mathbb{C}_0} |\psi(s)| = 2^{-\varepsilon};$$

and, in particular, $\|2^{-j\varphi}\|_{\mathcal{A}^+} \underset{j\to+\infty}{\longrightarrow} 0$. Now, if n is any positive integer, let $j = j(n)$ be the integer such that $2^j \leqslant n < 2^{j+1}$, and set $r = n\, 2^{-j}$, so that $1 \leqslant r < 2$. By using (4), we get:

$$\begin{aligned} \|n^{-\phi}\|_{\mathcal{A}^+} = \|n^{-\varphi}\|_{\mathcal{A}^+} &= \|2^{-j\varphi} r^{-\varphi}\|_{\mathcal{A}^+} \leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} \|r^{-\varphi}\|_{\mathcal{A}^+} \\ &\leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} r^{\|\varphi\|_{\mathcal{A}^+}} \leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} 2^{\|\varphi\|_{\mathcal{A}^+}}. \end{aligned}$$

This shows that $\|n^{-\phi}\|_{\mathcal{A}^+} \underset{n\to\infty}{\longrightarrow} 0$ (more precisely, we have $\|n^{-\phi}\|_{\mathcal{A}^+} = \mathrm{O}\,(n^{-\delta})$ for some $\delta > 0$), and so C_ϕ is compact, by part *(b) (i)* of the theorem. □

Remark. Using the notation of Theorem 2, we have:

$$\|n^{-\phi}\|_{\mathcal{A}^+} = \|n^{-\varphi}\|_{\mathcal{A}^+},$$

and, in particular, the integer c_0 plays no role for the continuity or the compactness of the composition operator C_ϕ on $\mathcal{A}^+$. This is quite amazing, since c_0 intervenes decisively in the study of composition operators on the Hilbert space $\mathcal{H}^2$ of the square-summable Dirichlet series (so much so that Gordon and Hedenmalm [8] called it "*characteristic*").

Corollary 5 *Let $\phi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$. Then C_ϕ is bounded if $\mathcal{R}e\, c_1 \geqslant \sum_{n=2}^{\infty} |c_n|$, and is compact if $\mathcal{R}e\, c_1 > \sum_{n=2}^{\infty} |c_n|$.*

Proof. Let $\varphi_0 \in \mathcal{A}^+$ be defined by $\varphi_0(s) = \sum_{n=2}^{\infty} c_n n^{-s}$. For each positive integer N, we have: $N^{-\phi(s)} = (N^{c_0})^{-s} N^{-c_1} N^{-\varphi_0(s)}$, and so the inequality (4) with $r = N$ gives:

$$\|N^{-\phi}\|_{\mathcal{A}^+} = N^{-\mathcal{R}e\, c_1} \|N^{-\varphi_0}\|_{\mathcal{A}^+} \leqslant N^{-\mathcal{R}e\, c_1} N^{\|\varphi_0\|_{\mathcal{A}^+}} = N^{-\mathcal{R}e\, c_1 + \sum_{n=2}^{\infty} |c_n|};$$

thus $\|N^{-\phi}\|_{\mathcal{A}^+}$ is less than 1 in the first case, and tends to 0 in the second case. Theorem 3 ends the proof. □

Note that under the assumption of Corollary 5, $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is actually a contraction: $\|C_\phi\| \leqslant 1$.

2.2 Some specific examples

One of the main differences between the study of composition operators on $\mathcal{A}^+$ and those on $A^+(\mathbb{T})$ is the fact that the function $z \mapsto z$ does not belong to $\mathcal{A}^+$. Therefore, it is not clear that if C_ϕ is a composition operator on $\mathcal{A}^+$, we must have $\sum_n |c_n| < +\infty$. In some cases, it is however true. The next proposition contains a partial result of this type.

Recall (see [14]) that $(\lambda_j)_{j\geqslant 1}$ is a Sidon set if

$$\sum_{j=1}^{N} |a_j| \leqslant C_0 \sup_{t\in\mathbb{R}} \Big| \sum_{j=1}^{N} a_j e^{i\lambda_j t} \Big|$$

for some finite positive constant C_0.

Proposition 6

(a) *If $2 \leqslant q_1 < q_2 < \cdots$ are multiplicatively independent integers and $\phi(s) = c_0 s + c_1 + \sum_{j=1}^{\infty} d_j q_j^{-s}$, then the boundedness of $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ implies $\mathcal{R}e\, c_1 \geqslant \sum_{j=2}^{\infty} |d_j|$, and its compactness implies $\mathcal{R}e\, c_1 > \sum_{j=2}^{\infty} |d_j|$.*

(b) *Let $(\lambda_j)_{j\geqslant 1}$ be a Sidon set of positive integers, r an integer $\geqslant 2$, and $\phi(s) = c_0 s + \varphi(s)$, where $\varphi \in \mathcal{D}$ and $\varphi(s) = c_1 + \sum_{j=1}^{\infty} d_j r^{-\lambda_j s}$ for $\mathcal{R}e\, s$ large. Then the boundedness of $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ requires that $\sum_{j=1}^{\infty} |d_j| < +\infty$.*

Proof. *(a)* Write $\varphi_0(s) = \sum_{j=1}^{\infty} d_j q_j^{-s}$, as in the proof of Corollary 5. For every integer $n \geqslant 2$, we have, for $\mathcal{R}e\, s$ large enough:

$$n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1} \exp\big(-\varphi_0(s)\log n\big) = (n^{c_0})^{-s} n^{-c_1} \sum_{k=0}^{\infty} \frac{(-\log n)^k}{k!}\, \varphi_0(s)^k.$$

Since C_ϕ is assumed to be bounded on $\mathcal{A}^+$, we know that $n^{-\phi} \in \mathcal{A}^+$, and so:

$$n^{-\phi(s)} = \sum_{j=1}^{\infty} a_{n,j} j^{-s}, \quad \text{with } \sum_{j=1}^{\infty} |a_{n,j}| < +\infty.$$

But the supports (spectra) of the φ_0^k's do not intersect: in fact, the spectrum of φ_0^k only involves finite products $\prod_j q_j^{\alpha_j}$, where $\sum \alpha_j = k$, and these products are all distinct. In particular, for $k = 1$, $(-\log n)\varphi_0(s)$ is part of the expansion of $n^{-\phi(s)}$, which means that $(-d_j \log n)_j$ is a subsequence of $(a_{n,j})_j$. Therefore, $\sum_j |d_j| < +\infty$ (and so $\varphi_0 \in \mathcal{A}^+$), and the series expansion of $\varphi_0(s)$ holds for every $s \in \mathbb{C}_0$.

Finally, since the $\log q_j$'s are rationally independent, Kronecker's Approximation Theorem implies that, for each $\sigma > 0$, we have:

$$\inf_{t\in\mathbb{R}} \mathcal{R}e\, \phi(\sigma + it) = c_0\sigma + \mathcal{R}e\, c_1 - \sum_{j=1}^{\infty} |d_j| q_j^{-\sigma}.$$

Since the left-hand side is $\geqslant 0$ by the first part of Theorem 3, we get $\mathcal{R}e\, c_1 \geqslant \sum_{j=1}^{\infty} |d_j|$, by letting σ go to zero. The compact case is similar.

(b) We have (see [17]):

$$\inf_{\tau\in\mathbb{R}} \sum_{j=1}^{N} \rho_j \cos(\lambda_j \tau + \xi_j) \leqslant -\delta \sum_{j=1}^{N} \rho_j \tag{5}$$

for some other constant $\delta > 0$, where the ρ_j's (non-negative) and the (real) ξ_j's are arbitrary. Without loss of generality, we can assume that $r = 2$. Fix an integer $J \geqslant 1$, and let $B : \mathbb{R} \to \mathbb{R}^+$ (see [15], p.165) be a non-negative Dirichlet polynomial (of the form $\sum \alpha_k e^{i\beta_k t}$, $\beta_k \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$) such that:

$$\widehat{B}(0) = \widehat{B}(\lambda_j \log 2) = 1, \quad 1 \leqslant j \leqslant J \tag{6}$$

(recall that $\widehat{B}(\lambda) = \lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} B(t) e^{-i\lambda t}\, dt$).

For large $\sigma > 0$, we have an absolutely convergent expansion:

$$\varphi(\sigma + i(t+\tau)) = c_1 + \sum_{j=1}^{\infty} d_j 2^{-\lambda_j \sigma} 2^{-\lambda_j it} e^{-i\lambda_j \tau \log 2},$$

so that, for $\mathcal{R}e\, s$ large enough (say $\mathcal{R}e\, s \geqslant \sigma_0 > 0$):

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(s + i\tau) B(\tau)\, d\tau = c_1 + \sum_{j=1}^{\infty} d_j 2^{-\lambda_j s} \widehat{B}(\lambda_j \log 2). \tag{7}$$

Actually, (7) holds for every s with positive real part σ. To see this, set:

$$f_T(s) = \frac{1}{2T} \int_{-T}^{T} \varphi(s + i\tau) B(\tau)\, d\tau.$$

Proposition 4 shows that $\mathcal{R}e\, \varphi(s+i\tau) > 0$ for $s \in \mathbb{C}_0$, and thus that $\mathcal{R}e\, f_T(s) > 0$ for $s \in \mathbb{C}_0$. Moreover, f_T as well as the right-hand side of (7), since B is a

Dirichlet polynomial, are holomorphic in $\mathbb{C}_0$; hence a normal family argument gives the above statement.

Therefore, if we take the real part of both sides of (7), we get, for every $\sigma > 0$ and t $\in \mathbb{R}$:

$$\mathcal{R}e\, c_1 + \sum_{j\geqslant 1} 2^{-\lambda_j \sigma} \mathcal{R}e \left(d_j 2^{-\lambda_j it} \widehat{B}(\lambda_j \log 2) \right) = \lim_{T\to+\infty} \mathcal{R}e\, f_T(\sigma + it) \geqslant 0.$$

Letting σ tend to zero gives:

$$\mathcal{R}e\, c_1 + \sum_{j\geqslant 1} \mathcal{R}e \left(d_j 2^{-\lambda_j it} \widehat{B}(\lambda_j \log 2) \right) \geqslant 0, \quad \text{for any } t \in \mathbb{R}.$$

Taking the infimum with respect to t and using (5), we get:

$$\mathcal{R}e\, c_1 - \delta \sum_{j=1}^{\infty} |d_j|\ |\widehat{B}(\lambda_j \log 2)| \geqslant 0$$

and therefore, using (6):

$$\mathcal{R}e\, c_1 - \delta \sum_{j=1}^{J} |d_j| \geqslant 0.$$

It follows that $\sum_{j=1}^{\infty} |d_j| \leqslant \frac{1}{\delta} \mathcal{R}e\, c_1$, and this ends the proof of Proposition 6. $\square$

Remark. The above proof gives the following information about Dirichlet series, which is actually not connected to composition operators: *let φ be a Dirichlet series which can be written as $\varphi(s) = c_1 + \sum_{j\geqslant 1} d_j r^{-\lambda_j s}$, where $(\lambda_j)_j$ is a Sidon sequence; if there is a $\beta \in \mathbb{R}$ such that $\varphi(\mathbb{C}_0) \subseteq \mathbb{C}_\beta$, then $\sum_{j\geqslant 1} |d_j| < +\infty$.*

However, in general, conditions like $\sum_{n\geqslant 2} |c_n| \leqslant \mathcal{R}e\, c_1$ (resp. $< \mathcal{R}e\, c_1$) are not necessary to have boundedness or compactness of the composition operator $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ (with $\phi(s) = c_0 s + c_1 + \sum_{n\geqslant 2} c_n n^{-s}$), as shown by the following examples.

Proposition 7 *Let $\phi(s) = c_0 s + c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$, where $r \geqslant 2$ and c_r, c_{r^2} are > 0. Then:*

(a) If we have

$$\mathcal{R}e\, c_1 > \frac{(c_r)^2}{8 c_{r^2}} + c_{r^2}, \tag{8}$$

$C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ *is bounded and even compact.*

(b) *Conversely, if* $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *is bounded, and moreover* $c_r \leqslant 4c_{r^2}$*, we must have*

$$\mathcal{R}e\, c_1 \geqslant \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}. \tag{9}$$

In fact, we must have (8) whenever C_ϕ *is compact.*

(c) *If*

$$\mathcal{R}e\, c_1 = \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}, \tag{10}$$

then $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *is bounded if and only if* $c_r \neq 4c_{r^2}$*.*

Proof. *(a)* and *(b)* follow immediately from Theorem 3, since (8) implies $\mathcal{R}e\,\phi(s) > c_0\mathcal{R}e\, s + \delta \geqslant \delta$ for every $s \in \mathbb{C}_0$, with $\delta = \mathcal{R}e\, c_1 - \left[\frac{(c_r)^2}{8c_{r^2}} + c_{r^2}\right]$, and, under the assumption that $c_r \leqslant 4c_{r^2}$, the converse is true. □

However, we shall give another proof, because we think that it sheds additional light.

Second proof.
(a) Without loss of generality, we may and shall assume that $r = 2$. We will make use (see [16, p. 60]) of the Hermite polynomials $H_0, H_1, \ldots$ defined by:

$$H_k(\lambda) = (-1)^k e^{\lambda^2} \frac{d^k}{d\lambda^k}\left(e^{-\lambda^2}\right) = (2\lambda)^k + \text{terms of lower degree.} \tag{11}$$

The exponential generating function of the H_k's is:

$$\sum_{k=0}^{\infty} \frac{H_k(\lambda)}{k!} x^k = \exp\left(2\lambda x - x^2\right). \tag{12}$$

Following Indritz [13], we have the sharp estimate

$$|H_k(\lambda)| \leqslant (2^k k!)^{1/2} e^{\lambda^2/2}, \tag{13}$$

for each $k \in \mathbb{N}_0$ and each $\lambda \in \mathbb{R}$. The estimate (13) implies the following:

Lemma 8 *Let* λ *be a real number, and* x *be a non-negative real number. Then we have:*

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C(1+x)^{1/2} \exp\left(x^2 + \frac{\lambda^2}{2}\right) \tag{14}$$

where C *is a positive constant.*

Proof of the Lemma. (13) implies that:

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant \sum_{k=0}^{\infty} \frac{(x\sqrt{2})^k}{(k!)^{1/2}} e^{\lambda^2/2}.$$

We now make use of the classical estimate (see *e.g.* Dieudonné [3, p. 195]):

$$\sum_{k=0}^{\infty} \frac{y^k}{(k!)^p} \sim \frac{1}{\sqrt{p}} (2\pi)^{\frac{1-p}{2}} y^{\frac{1-p}{2p}} \exp(py^{1/p}) \quad \text{as } y \to \infty \ (p > 0 \text{ fixed}). \tag{15}$$

Using the above, with $p = \frac{1}{2}$ and $y = x\sqrt{2}$, we obtain for some constant C :

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C e^{\lambda^2/2} (1+x)^{1/2} e^{x^2},$$

proving the Lemma. □

Note that, if we wish to avoid the use of (15), we can easily obtain the slightly weaker estimate:

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C_a \exp\left(ax^2 + \frac{\lambda^2}{2}\right), \quad \text{for each } a > 1. \tag{16}$$

Indeed, we have by the Cauchy-Schwarz inequality and by (13) :

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k &= \sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{(k!)^{1/2}(2a)^{k/2}} \frac{(2a)^{k/2} x^k}{(k!)^{1/2}} \\
&\leqslant \left(\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|^2}{k!(2a)^k}\right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{(2a)^k x^{2k}}{k!}\right)^{1/2} \\
&\leqslant e^{\lambda^2/2} \Big(\sum_{k=0}^{\infty} a^{-k}\Big)^{1/2} \exp(ax^2) \\
&= (1-a^{-1})^{-1/2} \exp\left(ax^2 + \frac{\lambda^2}{2}\right).
\end{aligned}$$

We now finish the proof of Proposition 7. First, we notice that:

$$n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1} \exp\left(-c_2 2^{-s} \log n - c_4 4^{-s} \log n\right).$$

We then set:

$$x_n = \sqrt{c_4 \log n}, \quad \lambda_n = \frac{-c_2}{2\sqrt{c_4}} \sqrt{\log n}, \quad x = 2^{-s} x_n, \tag{17}$$

which allows us to write $n^{-\phi(s)}$ under the form:

$$n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1} \exp\left(2\lambda_n x - x^2\right) = (n^{c_0})^{-s} n^{-c_1} \sum_{k=0}^{\infty} \frac{H_k(\lambda_n)}{k!} x_n^k (2^k)^{-s}.$$

This implies that we have the equality:

$$\|n^{-\phi}\|_{\mathcal{A}^+} = n^{-\mathcal{R}e\, c_1} \sum_{k=0}^{\infty} \frac{|H_k(\lambda_n)|}{k!} x_n^k. \tag{18}$$

If we now use Lemma 8 and change C (if necessary), we get for $n \geqslant 2$:

$$\begin{aligned}\|n^{-\phi}\|_{\mathcal{A}^+} &\leqslant C n^{-\mathcal{R}e\, c_1} (\log n)^{1/4} \exp\big(x_n^2 + \frac{\lambda_n^2}{2}\big) \\ &= C(\log n)^{1/4} n^{-\mathcal{R}e\, c_1} n^{\frac{c_2^2}{8c_4} + c_4} =: \epsilon_n.\end{aligned}$$

By (8), we have $\epsilon_n \to 0$, implying that $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact as a consequence of Theorem 3.

(b) The identities (18) and (12) imply that we have, for each real θ,

$$\begin{aligned} n^{\mathcal{R}e\, c_1} \|n^{-\phi}\|_{\mathcal{A}^+} &\geqslant \left| \sum_{k=0}^{\infty} \frac{H_k(\lambda_n)}{k!} x_n^k e^{ik\theta} \right| = \left|\exp\big(2\lambda_n x_n e^{i\theta} - x_n^2 e^{2i\theta}\big)\right| \\ &= \exp\big(2\lambda_n x_n \cos\theta - x_n^2 \cos 2\theta\big). \end{aligned}$$

Setting $t = \cos\theta$, we see that

$$2\lambda_n x_n \cos\theta - x_n^2 \cos 2\theta = 2\lambda_n x_n t - x_n^2(2t^2 - 1)$$

is maximum for $t = \frac{\lambda_n}{2x_n} = \frac{-c_2}{4c_4}$, and this t will be admissible if $\left|\frac{-c_2}{4c_4}\right| \leqslant 1$, *i.e.* if $c_2 \leqslant 4c_4$ (recall that c_2, c_4 are positive). For this value of t, we get:

$$\|n^{-\phi}\|_{\mathcal{A}^+} \geqslant n^{-\mathcal{R}e\, c_1 + \frac{c_2^2}{8c_4} + c_4}, \quad n = 1, 2, \dots \quad , \quad c_2 \leqslant 4c_4. \tag{19}$$

Now, if C_ϕ is bounded, $\|n^{-\phi}\|_{\mathcal{A}^+}$ is bounded from above, and (19) implies that $\mathcal{R}e\, c_1 \geqslant \frac{(c_2)^2}{8c_4} + c_4$. If C_ϕ is compact, $\|n^{-\phi}\|_{\mathcal{A}^+} \to 0$ and (19) implies that $\mathcal{R}e\, c_1 > \frac{(c_2)^2}{8c_4} + c_4$. □

Remark. Condition (8) is a more general sufficient condition for the boundedness of C_ϕ than the "*trivial*" sufficient condition $\mathcal{R}e\, c_1 \geqslant |c_2| + |c_4|$ of Corollary 5 if and only if $c_r < 8c_{r^2}$. This might be due to the highly oscillatory character of the Hermite polynomials $H_k(\lambda)$, involving a term $\cos\big(\sqrt{2k+1}\lambda - k\frac{\pi}{2}\big)$ (see [16, p. 67]), which we ignore when we majorize $|H_k(\lambda)|$ as in (13).

End of proof of Proposition 7. *(c)* We still assume that $r = 2$. First, if $c_2 > 4c_4$, then (10) implies that $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ for some $\delta > 0$, and we are done. So, we assume that $c_2 \leqslant 4c_4$. We have:

$$\begin{aligned}\|(2^j)^{-\phi}\|_{\mathcal{A}^+} &= \|2^{-j\phi}\|_{\mathcal{A}^+} = \|(2^{-c_1 - c_2 2^{-s} - c_4 4^{-s}})^j\|_{\mathcal{A}^+} \\ &= \big\|\big(\exp[(-c_1 - c_2 2^{-s} - c_4 4^{-s})\log 2]\big)^j\big\|_{\mathcal{A}^+} = \|\psi^j\|_{A^+(\mathbb{T})},\end{aligned}$$

with

$$\psi(z) = \exp\big(-(c_1 + c_2 z + c_4 z^2)\log 2\big).$$

We then apply Newman's result (quoted as (a) in the Introduction: see [20]) to check whether the sequence $(\|\psi^j\|_{A^+(\mathbb{T})})_j$ is bounded. Let $\theta_0 \in [0, 2\pi[$ be such that $|\psi(e^{i\theta_0})| = 1$. We look for the coefficient of t^2 in the Taylor expansion of:

$$\log\psi(e^{i\theta_0+it}) = -(c_1 + c_2 e^{i\theta_0}e^{it} + c_4 e^{2i\theta_0}e^{2it})\log 2.$$

This term is:

$$\Big(\frac{c_2}{2}e^{i\theta_0} + 2c_4 e^{2i\theta_0}\Big)\log 2,$$

and its real part is:

$$\Big(\frac{c_2}{2}\cos\theta_0 + 2c_4(2\cos^2\theta_0 - 1)\Big)\log 2. \tag{20}$$

Now, remark that the condition $|\psi(e^{i\theta_0})| = 1$ means that:

$$\mathcal{R}e\, c_1 = -c_2\cos\theta_0 - c_4(2\cos^2\theta_0 - 1),$$

which gives, using (10), $(4c_4\cos\theta_0 + c_2)^2 = 0$, that is $\cos\theta_0 = -\frac{c_2}{4c_4}\cdot$ Hence (20) is equal to 0 if and only if $c_2 = 4c_4$.

But in this case, $\theta_0 = \pi$, and Taylor's expansion becomes:

$$\log\psi(e^{i(\theta_0+t)}) = d_1 + d_2 t + 0.t^2 + i\log 2\frac{2c_4}{3}t^3 + \cdots$$

Hence, in Newman's terminology (see [20], and see (a) in the Introduction), the point $e^{i\theta_0}$ is not an ordinary point, and so the sequence $(\|\psi^j\|_{A^+(\mathbb{T})})_j$ is not bounded. It follows that the sequence $(\|2^{-j\phi}\|_{\mathcal{A}^+})_j$ is not bounded either.

In the case $c_2 < 4c_4$, the point $e^{i\theta_0}$ is ordinary, and so $(\|2^{-j\phi}\|_{\mathcal{A}^+})_j$ is bounded. Since $\sum_n |c_n| = |c_1| + |c_2| + |c_4| < +\infty$, the argument used in the proof in Theorem 3, *(b) (ii)* gives the boundedness of C_ϕ. □

Remark. Part *(c)* of Proposition 7 shows that, if $\mathcal{R}e\, c_1 = \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}$ and $c_r = 4c_{r^2}$ (so $\mathcal{R}e\, c_1 = 3c_4$, and $\psi(s) = ia + c(3 + 4\cdot 2^{-s} + 4^{-s})$, with $a \in \mathbb{R}$ and $c > 0$), then C_ϕ is not bounded on $\mathcal{A}^+$, though $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_0$ (and $\sum_n |c_n| < +\infty$).

3 Automorphisms of $A^+(\mathbb{T}^k)$, $A^+(\mathbb{T}^\infty)$, $\mathcal{A}^+$

In this section, we will make repeated use of the following Lemma (see (b) of the Introduction):

Lemma 9 *Let $\phi(z) = \prod_{j=1}^{J} \epsilon_j \frac{z-a_j}{1-\overline{a_j}z}$, where $|\epsilon_j| = 1$ and $a_j \in \mathbb{D}$. Suppose that $\|\phi^n\|_{A^+}$ remains bounded ($n = 1, 2, \ldots$). Then, $a_j = 0$ for each j.*

Proof. This Lemma is well-known (see [20] page 38, assertion (3),or [14], page 77). For example, if $a_j \neq 0$ for some j, we have $\phi(e^{it}) = e^{ig(t)}$, where g is a $\mathcal{C}^2$, real, non affine function; and the Van der Corput inequalities show that we even have: $\|\phi^n\|_{A^+} \geqslant \delta\sqrt{n}$. □

Since $|\phi(e^{it})| = 1$, Lemma 9 can be viewed as a special case of the following Lemma (which will be needed only in Section 4, but which we state here because it is the natural extension of Lemma 9), due to Beurling and Helson, and this Lemma is itself a special case of Cohen's Theorem (see [22], page 93, corollary of Theorem 4.7.3). We shall use the following definition:

Let G be a discrete abelian group, and Γ be its (compact) dual group; the Wiener algebra $A(\Gamma)$ is the set of functions $f : \Gamma \to \mathbb{C}$ which can be written as an absolutely convergent series $f(\gamma) = \sum_1^\infty a_n(x_n, \gamma)$, with the norm $\|f\|_{A(\Gamma)} = \sum_1^\infty |a_n|$, and where (x_n, γ) denotes the action of $\gamma \in \Gamma$ on the element x_n of G. We are now ready to state:

Lemma 10 *(Beurling-Helson). Let G be a discrete abelian group, with* connected *dual group Γ. Let $\phi \in A(\Gamma)$, which does not vanish on Γ, and such that $\|\phi^n\|_{A(\Gamma)} \leqslant C$ for some constant C $(n = 0, \pm 1, \pm 2, \ldots)$. Then, ϕ is affine, i.e. there exist a complex number a with $|a| = 1$ and an element x of G such that $\phi(\gamma) = a(x, \gamma)$ for any $\gamma \in \Gamma$.*

Let us now consider the Wiener algebra $A^+(\mathbb{T}^k)$ in k variables, *i.e.* the algebra of functions $f : \overline{\mathbb{D}}^k \to \mathbb{C}$ which can be written as:

$$f(z) = \sum_{n_1,\ldots,n_k \geqslant 0} a(n_1, \ldots, n_k) z_1^{n_1} \ldots z_k^{n_k}, \quad z = (z_1, \ldots, z_k),$$

with the norm $\|f\|_{A^+(\mathbb{T}^k)} = \sum\limits_{n_1,\ldots,n_k \geqslant 0} |a(n_1, \ldots, n_k)| < +\infty$.

If $\phi = (\phi_1, \ldots, \phi_k) : \mathbb{D}^k \to \mathbb{C}^k$ is an analytic function, the composition operator C_ϕ will be bounded on $A^+(\mathbb{T}^k)$ if and only if :

$$\|\phi_j^n\|_{A^+(\mathbb{T}^k)} \leqslant C, \qquad j = 1, \ldots, k, \text{ and } \quad n = 0, 1, 2, \ldots \tag{21}$$

(the proof is the same as in Newman's case $k = 1$).

Then, since $\|\phi_j\|_\infty = \lim\limits_{n\to\infty} \|\phi_j^n\|_{A^+(\mathbb{T}^k)}^{1/n}$, we see that ϕ necessarily maps $\mathbb{D}^k$ into $\overline{\mathbb{D}}^k$. We can now state:

Theorem 11 *Assume that the map $\phi \colon \mathbb{D}^k \to \overline{\mathbb{D}}^k$ induces a bounded operator $C_\phi \colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$. Then C_ϕ is an automorphism of $A^+(\mathbb{T}^k)$ if and only if $\phi(z) = (\epsilon_1 z_{\sigma(1)}, \ldots, \epsilon_k z_{\sigma(k)})$ for some permutation σ of $\{1, \ldots, k\}$ and some complex signs $\epsilon_1, \ldots, \epsilon_k$.*

Proof. The sufficient condition is trivial. For the necessary one, we first observe that, for $j = 1, \ldots, k$, $\phi_j \in A^+(\mathbb{T}^k)$, since $\phi_j = C_\phi(z_j)$; hence ϕ can be continuously extended to a continuous map, still denoted by ϕ, from $\overline{\mathbb{D}}^k$ to $\overline{\mathbb{D}}^k$.

We are going to show that this map is bijective.
Assume first that $a, b \in \overline{\mathbb{D}}^k$ and that $\phi(a) = \phi(b)$. Let $f \in A^+(\mathbb{T}^k)$; since C_ϕ is bijective we can find $g \in A^+(\mathbb{T}^k)$ such that $f = g \circ \phi$, so that $f(a) = f(b)$. Since $A^+(\mathbb{T}^k)$ obviously separates the points of $\overline{\mathbb{D}}^k$, we have $a = b$. In particular, ϕ is injective on $\mathbb{D}^k$ and by Osgood's Theorem (see [19], Theorem 5, page 86) $\det(\phi'(z)) \neq 0$ for each $z \in \mathbb{D}^k$, implying that ϕ is an open mapping on $\mathbb{D}^k$. Therefore, $\phi(\mathbb{D}^k) \subseteq \mathbb{D}^k$.
Now, let $u \in \overline{\mathbb{D}}^k$. Define an element L of the spectrum of $A^+(\mathbb{T}^k)$ by $L(f) = g(u)$ if $f = g \circ \phi$. Since the spectrum of $A^+(\mathbb{T}^k)$ is clearly $\overline{\mathbb{D}}^k$, we can find $a \in \overline{\mathbb{D}}^k$ such that $L(f) = f(a)$, so that $g\big(\phi(a)\big) = g(u)$ for any $g \in A^+(\mathbb{T}^k)$, implying $u = \phi(a)$. ϕ is therefore a homeomorphism : $\overline{\mathbb{D}}^k \to \overline{\mathbb{D}}^k$.

Since $\phi(\overline{\mathbb{D}}^k) = \overline{\mathbb{D}}^k$ and $\phi(\mathbb{D}^k) \subseteq \mathbb{D}^k$, we get $\phi(\mathbb{D}^k) = \mathbb{D}^k$. In particular, $\phi \in \mathrm{Aut}\, \mathbb{D}^k$, the group of analytic automorphisms of $\mathbb{D}^k$.

Recall that ([19], Proposition 3, page 68):

Lemma 12 *The analytic map $\phi \colon \mathbb{D}^k \to \mathbb{D}^k$ belongs to* $\mathrm{Aut}\, \mathbb{D}^k$ *if and only if*

$$\phi(z) = \left(\epsilon_1 \frac{z_{\sigma(1)} - a_1}{1 - \overline{a_1} z_{\sigma(1)}}, \ldots, \epsilon_k \frac{z_{\sigma(k)} - a_k}{1 - \overline{a_k} z_{\sigma(k)}} \right),$$

for some permutation σ of $\{1, \ldots, k\}$, for some $(a_1, \ldots, a_k) \in \mathbb{D}^k$ and some complex signs $\epsilon_1, \ldots, \epsilon_k$.

We therefore see that $\phi_j(z) = \epsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a_j} z_{\sigma(j)}}$, so that for each $n \in \mathbb{N}$, we have in view of (21):

$$\left\| \left(\epsilon_j \frac{z - a_j}{1 - \overline{a_j} z} \right)^n \right\|_{A^+} = \left\| \left(\epsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a_j} z_{\sigma(j)}} \right)^n \right\|_{A^+(\mathbb{T}^k)} \leqslant C.$$

Lemma 9 now implies that $a_j = 0$, $j = 1, \ldots, k$, so that $\phi_j(z) = \epsilon_j z_{\sigma(j)}$, and this ends the Proof of Theorem 11. $\square$

We now consider the Wiener algebra $A^+(\mathbb{T}^\infty)$ in countably many variables. It will be convenient to consider holomorphic functions on the open unit ball $\mathbf{B} = \mathbb{D}^\infty \cap \mathbf{c}_0$ of the Banach space $\mathbf{c}_0$ of sequences $z = (z_n)_{n \geqslant 1}$ tending to zero at infinity, with its natural norm $\|z\| = \sup_{n \geqslant 1} |z_n|$. We then have the following extension of Cartan's Lemma 12 to the case of $\mathbf{B}$, which is due to Harris ([9]):

Lemma 13 (Analytic Banach-Stone Theorem). *The analytic automorphisms $\phi \colon \boldsymbol{B} \to \boldsymbol{B}$ are exactly the maps of the form $\phi = (\phi_j)_{j \geqslant 1}$, with $\phi_j(z) = \epsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a_j} z_{\sigma(j)}}$, for some permutation σ of $\mathbb{N}$, some point $a = (a_j)_{j \geqslant 1} \in \boldsymbol{B}$, and some sequence $(\epsilon_j)_{j \geqslant 1}$ of complex signs.*

Recall that the linear Banach-Stone Theorem states : if $L \colon \mathbf{c}_0 \to \mathbf{c}_0$ is a surjective isometry fixing the origin, then L has the form:

$$L(z_1, \ldots, z_n, \ldots) = (\epsilon_1 z_{\sigma(1)}, \ldots, \epsilon_n z_{\sigma(n)}, \ldots).$$

If we want to exploit Lemma 13 for describing the composition automorphisms of $A^+(\mathbb{T}^\infty)$, we have to make an extra assumption, the reason for which is the following: if C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$, then ϕ is an automorphism of $\overline{\mathbb{D}}^\infty$, but there is no reason, *a priori*, why ϕ should be an automorphism of $\mathbf{B}$.

Theorem 14 *Let $\phi = (\phi_j)_j \colon \boldsymbol{B} \to \boldsymbol{B}$ be an analytic map such that C_ϕ maps $A^+(\mathbb{T}^\infty)$ into itself. Then :*

- *(a) If $\phi(z) = (\epsilon_j z_{\sigma(j)})_{j\geqslant 1}$ for some permutation σ of $\mathbb{N}$ and some sequence $(\epsilon_j)_{j\geqslant 1}$ of complex signs, then C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$, and it is isometric.*
- *(b) If C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$* and if we moreover *assume that $\phi_k(z) = z_k^{d_k} u_k(z)$, with $d_k \geqslant 1$ and $u_k(0) \neq 0$, for each $k \in \mathbb{N}$ and each $z \in \boldsymbol{B}$, then $\phi(z) = (\epsilon_j z_j)_{j\geqslant 1}$ for some sequence $(\epsilon_j)_{j\geqslant 1}$ of complex signs.*

Proof. *(a)* is trivial. For *(b)*, consider the compact set $K = \overline{\mathbb{D}}^\infty$, endowed with the product topology (K is nothing but the spectrum of $A^+(\mathbb{T}^\infty)$); clearly, $\mathbf{B}$ is dense in K, and since $\phi_j = C_\phi(z_j) \in A^+(\mathbb{T}^\infty)$, $\phi_j \colon \mathbf{B} \to \mathbb{D}$ extends continuously to K, and $\phi = (\phi_j)_j$ extends continuously to a map, still denoted by ϕ, from K to K, and we still can write, for every $k \in \mathbb{N}$, $\phi_k(z) = z_k^{d_k} u_k(z)$ for each $z \in K$. Exactly as in the Proof of Theorem 11, we can show that ϕ is bijective, since K is the spectrum of $A^+(\mathbb{T}^\infty)$. Let now $\psi \colon K \to K$ be the inverse map of ϕ. Since K is compact, ψ is continuous on K, and so on $\mathbf{B}$; it is then easy to see, as usual, that ψ is holomorphic in $\mathbf{B}$ (alternatively, $\psi_k = (C_\phi)^{-1}(z_k) \in A^+(\mathbb{T}^\infty)$, and so is analytic in $\mathbb{D}^\infty$, and it is clear that $\psi = (\psi_k)_k$).

Now, it suffices to show that ψ maps $\mathbf{B}$ into $\mathbf{B}$; indeed, it will follow that ϕ maps $\mathbf{B}$ onto $\mathbf{B}$, and so the map ϕ will appear as an analytic automorphism of $\mathbf{B}$ (since we already know that $\psi = \phi^{-1}$ is analytic in $\mathbf{B}$), and Lemma 13 shows that $\phi_j(z)$ has the form $\epsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a_j} z_{\sigma(j)}}\cdot$ Now, $\|\phi_j^n\|_{A^+(\mathbb{T}^\infty)} = \|C_\phi(z_j^n)\|_{A^+(\mathbb{T}^\infty)} \leqslant \|C_\phi\|$, and as in the Proof of Theorem 11, we shall conclude that $a_j = 0$ for each j. Finally, the assumption $\phi_k(z) = z_k^{d_k} u_k(z)$ for each k will imply that σ is the identity map.

So we have to show that $\psi(\mathbf{B}) \subseteq \mathbf{B}$. If it were not the case, it would exist an element $w = (w_j)_j \in \mathbf{B}$ such that $\psi(w) \notin \mathbf{B}$. Hence there would exist $\delta > 0$ and an infinite subset $J \subseteq \mathbb{N}$ such that

$$|\psi_j(w)| > \delta \text{ for every } j \in J. \tag{22}$$

Let $\delta' = \delta / \|C_\psi\|$.

Since $w \in \mathbf{B}$, we should find an integer $N \geqslant 1$ such that

$$n \geqslant N \quad \Rightarrow \quad |w_n| \leqslant \delta'.$$

Let $\kappa = \max_{1\leqslant n\leqslant N} |w_n|$. Since $\kappa < 1$, there would exist $p \geqslant 1$ such that $\kappa^p < \delta'$. Consider the finite set:

$$F = \{\alpha = (m_1, \ldots, m_N, 0, \ldots)\,;\ m_1 + \cdots + m_N \leqslant p\}.$$

We assert that:

$$F \text{ intersects the spectrum of } \psi_j \text{ for every } j \in J. \tag{23}$$

Indeed, writing:

$$\psi_j(z) = \sum a_j(n_1, \ldots, n_k, 0, \ldots) z_1^{n_1} \ldots z_k^{n_k},$$

we have:

- if $\alpha = (n_1, \ldots, n_l, \ldots)$ with $l > N$ and $n_l \neq 0$, then $|w_l| \leqslant \delta'$, and so:

$$|w^\alpha| \leqslant |w_1^{n_1} \ldots w_l^{n_l}| \leqslant |w_l^{n_l}| \leqslant |w_l| \leqslant \delta';$$

- if $n_1 + \cdots n_N \geqslant p$, then:

$$|w_1^{n_1} \cdots w_N^{n_N}| \leqslant \kappa^{n_1 + \cdots + n_N} \leqslant \kappa^p < \delta'.$$

Hence, in both cases, $\alpha \notin F$ implies $|w^\alpha| < \delta'$. Therefore, if F does not intersect the spectrum of ψ_j, we get:

$$|\psi_j(w)| \leqslant \sum_{\alpha \notin F} |a_j(\alpha)|\, |w^\alpha| \leqslant \delta' \|\psi_j\|_{A^+(\mathbb{T}^\infty)} \leqslant \delta' \|C_\psi\| = \delta$$

(since $\|\psi_j\|_{A^+(\mathbb{T}^\infty)} = \|C_\psi(z_j)\|_{A^+(\mathbb{T}^\infty)} \leqslant \|C_\psi\|\, \|z_j\|_{A^+(\mathbb{T}^\infty)} = \|C_\psi\|$), which contradicts (22).

To end the proof, remark now that the assumption $\phi_k(z) = z_k^{d_k} u_k(z)$ for every $k \in \mathbb{N}$ implies that:

$$z_k = \phi_k\big[\psi(z)\big] = [\psi_k(z)]^{d_k} u_k[\psi(z)].$$

But this is impossible, since J is infinite and, for $k \in J$, $\psi_k(z)$ depends on $(z_1, \ldots, z_N)$, and hence $\phi_k\big[\psi(z)\big] = [\psi_k(z)]^{d_k} u_k[\psi(z)]$ also (since $d_k \geqslant 1$ and $u_k(0) \neq 0$).

That ends the proof of Theorem 14. □

Remark. We shall see later, in Section 4, Theorem 21, that the converse of *(a)* in Theorem 14 is true.

Although Theorem 14 is not completely satisfactory, it will be sufficient for characterizing the composition automorphisms of the Wiener-Dirichlet algebra $\mathcal{A}^+$. In fact, we have:

Theorem 15 *Let $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ be a composition operator. Then C_ϕ is an automorphism of $\mathcal{A}^+$ if and only if ϕ is a vertical translation: $\phi(s) = s + i\tau$, where τ is a real number.*

Note that a similar result was obtained by F. Bayart [1] for the Hilbert space $\mathcal{H}^2$ of square-summable Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s}$ such that $\sum_1^\infty |a_n|^2 < +\infty$, but his proof does not seem to extend to our setting, and our strategy for proving Theorem 15 will be to deduce it from Theorem 14, with the help of the transfer operator Δ mentioned in the Introduction. The following Lemma (with the notation used in the Introduction) allows the transfer from composition operators on $\mathcal{A}^+$ to composition operators on $A^+(\mathbb{T}^\infty)$.

Lemma 16 *Suppose that $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is a composition operator, with $\phi(s) = c_0 s + \varphi(s)$, $c_0 \in \mathbb{N}_0$, $\varphi \in \mathcal{D}$. Let $T = \Delta C_\phi \Delta^{-1}\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$. Then:*

(a) $T = C_{\tilde\phi}$, where $\tilde\phi\colon \boldsymbol{B} \to \mathbb{D}^\infty$ is an analytic map such that $\tilde\phi(z^{[s]}) = z^{[\phi(s)]}$, for any $s \in \mathbb{C}_0$.

(b) If moreover $c_0 \geqslant 1$ (which is the case if C_ϕ is surjective), $\tilde\phi$ maps $\boldsymbol{B}$ into $\boldsymbol{B}$.

Proof. *(a)* Define $f_k(s) = p_k^{-\phi(s)} \in \mathcal{A}^+$, $\phi_k = \Delta f_k$ and

$$\tilde\phi = (\phi_1, \phi_2, \ldots). \tag{24}$$

We have

$$\tilde\phi(z^{[s]}) = (\Delta f_k(z^{[s]}))_{k\geqslant 1} = (f_k(s))_{k\geqslant 1} = z^{[\phi(s)]}$$

by (1), and $\|\phi_k\|_\infty = \|f_k\|_\infty \leqslant 1$ by (2). Moreover, no ϕ_k is constant, so the open mapping theorem implies that $|\phi_k(z)| < 1$ for $z \in \mathbf{B}$, *i.e.* $\tilde\phi(z) \in \mathbb{D}^\infty$. Finally, if $f(z) = \sum_{n=1}^\infty a_n z_1^{\alpha_1} \ldots z_r^{\alpha_r} \in A^+(\mathbb{T}^\infty)$ (where $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ is the decomposition in prime factors), we have the following "diagram":

$$f \overset{\Delta^{-1}}{\longmapsto} \sum_{n=1}^\infty a_n n^{-s} \overset{C_\phi}{\longmapsto} \sum_{n=1}^\infty a_n f_1^{\alpha_1} \ldots f_r^{\alpha_r} \overset{\Delta}{\longmapsto} \sum_{n=1}^\infty a_n \phi_1^{\alpha_1} \ldots \phi_r^{\alpha_r} = f \circ \tilde\phi,$$

i.e. $T(f) = C_{\tilde\phi}(f)$.

(b) First observe that C_φ also maps $\mathcal{A}^+$ into $\mathcal{A}^+$ (see the remark before Corollary 5). Secondly, we have $f_k(s) = p_k^{-c_0 s} p_k^{-\varphi(s)} = p_k^{-c_0 s} g_k(s)$, with $g_k \in \mathcal{A}^+$ and $\|g_k\|_{\mathcal{A}^+} = \|C_\varphi(p_k^{-s})\|_{\mathcal{A}^+} \leqslant C$. It follows that, for $z \in \mathbf{B}$: $\Delta f_k(z) = z_k^{c_0} \Delta g_k(z)$, and *via* (2) that:

$$|\Delta f_k(z)| \leqslant |z_k|^{c_0} \|\Delta g_k\|_\infty = |z_k|^{c_0} \|g_k\|_\infty \leqslant |z_k|^{c_0} \|g_k\|_{\mathcal{A}^+} \leqslant C |z_k|^{c_0}.$$

Since $c_0 \geqslant 1$, we see that $\Delta f_k(z) \to 0$ as $k \to \infty$, *i.e.* $\tilde\phi(z) \in \mathbf{B}$. Finally, whenever C_ϕ is surjective, $\phi : \mathbb{C}_0 \to \mathbb{C}_0$ is injective: indeed, $\mathcal{A}^+$ separates the points of $\mathbb{C}_0$ ($2^{-a} = 2^{-b}$ and $3^{-a} = 3^{-b}$ imply $a = b$, since $\log 2/\log 3$ is irrational), and we can argue as in Theorem 11.

To end the proof of Lemma 16, it remains to remark that if $c_0 = 0$, ϕ is never injective on $\mathbb{C}_0$, according to well-known results on the theory of analytic, almost-periodic functions (see *e.g.* Favard [4, p. 13]). Therefore, we have $c_0 \geqslant 1$ if C_ϕ is surjective. $\square$

Proof of Theorem 15. The sufficiency of the condition is trivial. Conversely, if C_ϕ is an automorphism of $\mathcal{A}^+$, let $C_{\tilde{\phi}} = \Delta C_\phi \Delta^{-1}$, as in Lemma 16. Since C_ϕ is surjective, we know from Lemma 16 that $\tilde{\phi}$ maps $\mathbf{B}$ into $\mathbf{B}$; we can apply Theorem 14, because $C_{\tilde{\phi}}$ is an automorphism of $A^+(\mathbb{T}^\infty)$ onto itself and moreover $\tilde{\phi}_k(z) = \Delta f_k(z) = z_k^{c_0} \Delta g_k(z)$, with $c_0 \geqslant 1$ (by Lemma 16 again) and

$$\Delta g_k(0) = \lim_{\mathcal{R}e\, s \to +\infty} g_k(s) = \lim_{\mathcal{R}e\, s \to +\infty} p_k^{-\varphi(s)} = p_k^{-c_1} \neq 0.$$

We conclude that:

$$\tilde{\phi}(z) = (\epsilon_1 z_1, \ldots, \epsilon_n z_n, \ldots), \tag{25}$$

for some sequence of signs $(\epsilon_n)_n$, where $z = (z_1, \ldots, z_n, \ldots)$.

If we now test this equality at the points $z^{[s]} = (p_j^{-s})_j$, $s \in \mathbb{C}_0$, and use (1), we see that

$$p_j^{-\phi(s)} = \epsilon_j p_j^{-s}, \quad s \in \mathbb{C}_0, \quad j \in \mathbb{N}. \tag{26}$$

Taking the moduli in (26), we get $\mathcal{R}e\, \phi(s) = \mathcal{R}e\, s$. Since $\phi(s) - s$ is analytic on the domain $\mathbb{C}_0$, this implies $\phi(s) - s = i\tau$, with $\tau \in \mathbb{R}$, thus ending the Proof of Theorem 15. $\square$

4 Isometries of $A^+(\mathbb{T}^k)$, $A^+(\mathbb{T}^\infty), \mathcal{A}^+$

In this section, we shall characterize the composition operators which are isometric on $A^+(\mathbb{T}^k)$ and then those which are isometric on $A^+(\mathbb{T}^\infty)$ (under an additional assumption) and on $\mathcal{A}^+$. If $f(z) = \sum a_\alpha z^\alpha \in A^+(\mathbb{T}^k)$, it will be convenient to note $a_\alpha = \widehat{f}(\alpha)$. The spectrum of f (denoted by $Sp\, f$) is the set of α's such that $\widehat{f}(\alpha) \neq 0$. e will denote the point $(1, \ldots, 1)$ of $\overline{\mathbb{D}}^k$. An elaboration of the method of Harzallah [14] allows us to show:

Theorem 17 *Assume that $\phi = (\phi_j)_j \colon \mathbb{D}^k \to \overline{\mathbb{D}}^k$, induces a composition operator $C_\phi \colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$. Then $C_\phi \colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$ is an isometry if and only if there exists a square matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant k}$, with $a_{ij} \in \mathbb{N}_0$ and* $\det A \neq 0$, *and complex signs $\epsilon_1, \ldots, \epsilon_k$ such that:*

$$\phi_i(z) = \epsilon_i z_1^{a_{i1}} \ldots z_k^{a_{ik}}, \quad 1 \leqslant i \leqslant k, \quad z = (z_1, \ldots, z_k) \in \mathbb{D}^k. \tag{27}$$

To prove this theorem, it will be convenient to use the following two Lemmas.

Lemma 18 *C_ϕ is an isometry if and only if:*

(a) $\phi_i = \epsilon_i F_i$, $1 \leqslant i \leqslant k$, where ϵ_i is a complex sign, $\widehat{F_i} \geqslant 0$, and $F_i(e) = \|F_i\|_\infty = 1$;

(b) if $\alpha, \alpha' \in \mathbb{N}_0^k$ are distinct, the spectra of ϕ^α and $\phi^{\alpha'}$ are disjoint.

Proof. Suppose that *(a)* and *(b)* hold, and let $f(z) = \sum \hat{f}(\alpha) z^\alpha \in A^+(\mathbb{T}^k)$. We have by *(b)*:

$$\|C_\phi f\|_{A^+(\mathbb{T}^k)} = \sum |\hat{f}(\alpha)| \, \|\phi^\alpha\|_{A^+(\mathbb{T}^k)} = \sum |\hat{f}(\alpha)| \, \|F^\alpha\|_{A^+(\mathbb{T}^k)},$$

since, with the obvious notation, $\phi^\alpha = \epsilon^\alpha F^\alpha$. Since $\widehat{F^\alpha} \geqslant 0$, we have, using *(a)* :

$$\|F^\alpha\|_{A^+(\mathbb{T}^k)} = F^\alpha(e) = 1,$$

so that:

$$\|C_\phi f\|_{A^+(\mathbb{T}^k)} = \sum |\hat{f}(\alpha)| = \|f\|_{A^+(\mathbb{T}^k)}.$$

Conversely, suppose that C_ϕ is an isometry. For each $i \in [1, k]$ and each $n \in \mathbb{N}$, we have $\|\phi_i^n\|_{A^+(\mathbb{T}^k)} = \|z_i^n\|_{A^+(\mathbb{T}^k)} = 1$, whence $\|\phi_i\|_\infty = \lim_{n\to\infty} \|\phi_i^n\|_{A^+(\mathbb{T}^k)}^{1/n} = 1$, by the spectral radius formula. Since $\|\phi_i\|_\infty \leqslant \|\phi_i\|_{A^+(\mathbb{T}^k)} = 1$, the only possibility is that $\phi_i = \epsilon_i F_i$, with $|\epsilon_i| = 1$, $\widehat{F_i} \geqslant 0$, and $\|\phi_i\|_{A^+(\mathbb{T}^k)} = 1 = \|F_i\|_{A^+(\mathbb{T}^k)} = F_i(e)$. Therefore, *(a)* holds. Now suppose that we can find $\alpha \neq \alpha'$ such that $Sp\,\phi^\alpha \cap Sp\,\phi^{\alpha'}$ contains an element $\beta_0 \in \mathbb{N}_0^k$, and set $\rho = \widehat{\phi^\alpha}(\beta_0)$, $\rho' = \widehat{\phi^{\alpha'}}(\beta_0)$. Without loss of generality, we may assume that $|\rho| \geqslant |\rho'|$. Let θ be a complex sign such that $|\rho + \theta\rho'| = |\rho| - |\rho'|$. Then, we have $\|z^\alpha + \theta z^{\alpha'}\|_{A^+(\mathbb{T}^k)} = 2$, whereas

$$\begin{aligned}
\|C_\phi(z^\alpha + \theta z^{\alpha'})\|_{A^+(\mathbb{T}^k)} &= \|\phi^\alpha + \theta\phi^{\alpha'}\|_{A^+(\mathbb{T}^k)} \\
&= \sum_{\beta\neq\beta_0} |\widehat{\phi^\alpha}(\beta) + \theta\widehat{\phi^{\alpha'}}(\beta)| + |\rho + \theta\rho'| \\
&\leqslant \sum_{\beta\neq\beta_0} |\widehat{\phi^\alpha}(\beta)| + \sum_{\beta\neq\beta_0} |\widehat{\phi^{\alpha'}}(\beta)| + |\rho| - |\rho'| \\
&= 1 - |\rho| + 1 - |\rho'| + |\rho| - |\rho'| = 2(1 - |\rho'|) < 2,
\end{aligned}$$

contradicting the isometric character of C_ϕ. □

Lemma 19 *If $\phi = (\phi_i)_i$ and if one of the ϕ_i's is not a monomial, then we can find a pair of distinct elements α, $\alpha' \in \mathbb{N}_0^k$ such that the spectra of ϕ^α and $\phi^{\alpha'}$ intersect.*

Proof. To avoid awkward notation, we will assume that $k = 3$, but it will be clear that the reasoning works for any value of k. Since only the spectra of the ϕ_i's are involved, we can assume without loss of generality that we have:

$$\begin{aligned}
\phi_1(z) &= z_1^{s_1} z_2^{s_2} z_3^{s_3} + z_1^{t_1} z_2^{t_2} z_3^{t^3}, \quad \text{with } (s_1, s_2, s_3) \neq (t_1, t_2, t_3), \\
\phi_2(z) &= z_1^{u_1} z_2^{u_2} z_3^{u_3}, \\
\phi_3(z) &= z_1^{v_1} z_2^{v_2} z_3^{v_3}
\end{aligned}$$

(in short, $\phi_1(z) = z^s + z^t$; $\phi_2(z) = z^u$; $\phi_3(z) = z^v$).

If $\alpha = (a, b, c)$, the spectrum of $\phi^\alpha = (z^s + z^t)^a z^{bu} z^{cv}$ consists of the triples

$$\rho s_j + (a - \rho)t_j + bu_j + cv_j = \rho(s_j - t_j) + at_j + bu_j + cv_j,$$

with $j = 1, 2, 3$ and $0 \leqslant \rho \leqslant a$. Therefore, if $\alpha' = (a', b', c')$, the spectra of ϕ^α and $\phi^{\alpha'}$ will intersect if and only if we can find $0 \leqslant \rho \leqslant a$ and $0 \leqslant \rho' \leqslant a'$ such that:

$$\rho(s_j - t_j) + at_j + bu_j + cv_j = \rho'(s_j - t_j) + a't_j + b'u_j + c'v_j, \quad j = 1, 2, 3,$$

or equivalently:

$$(\rho - \rho')(s_j - t_j) + (a - a')t_j + (b - b')u_j = (c' - c)v_j, \quad j = 1, 2, 3. \tag{28}$$

In (28), we can drop the conditions $\rho \leqslant a$, $\rho' \leqslant a'$, since we can always replace a and a' by $a + N$ and $a' + N$, where N is a large integer, without affecting the result. Now, let M be the matrix:

$$M = \begin{bmatrix} s_1 - t_1 & t_1 & u_1 \\ s_2 - t_2 & t_2 & u_2 \\ s_3 - t_3 & t_3 & u_3 \end{bmatrix}.$$

To solve equation (28), we distinguish two cases.

Case 1 : $\det M = 0$.

We decide then to take $c' = c$. Since the field $\mathbb{Q}$ of rational numbers is the quotient field of $\mathbb{Z}$, we can find $\lambda, \mu, \nu \in \mathbb{Z}$, not all zero, such that:

$$\lambda(s_j - t_j) + \mu t_j + \nu u_j = 0, \quad j = 1, 2, 3.$$

If μ and ν are both zero, then $\lambda = 0$, since $s_j - t_j \neq 0$ for some j. Therefore, we may assume for example that $\mu \neq 0$, and write $\lambda = \rho - \rho'$, $\mu = a - a'$, $\nu = b - b'$, with $\alpha = (a, b, c) \in \mathbb{N}_0^3$, $\alpha' = (a', b', c') \in \mathbb{N}_0^3$, and $\alpha \neq \alpha'$ since $a \neq a'$. By construction, we have (28), so that the spectra of ϕ^α and $\phi^{\alpha'}$ are not disjoint.

Case 2 : $\det M \neq 0$.

We can then find rational numbers q, r, s such that:

$$q(s_j - t_j) + rt_j + su_j = v_j, \quad j = 1, 2, 3,$$

and we can write $q = \frac{\lambda}{N}$, $r = \frac{\mu}{N}$, $s = \frac{\nu}{N}$, where $\lambda, \mu, \nu \in \mathbb{Z}$ and where N is a positive integer. Therefore, we have:

$$\lambda(s_j - t_j) + \mu t_j + \nu u_j = Nv_j, \quad 1 \leqslant j \leqslant 3,$$

and writing $\lambda = \rho - \rho'$, $\mu = a - a'$, $\nu = b - b'$, $c = 0$, $c' = N$, we get (28) with distinct triples $\alpha = (a, b, c)$ and $\alpha' = (a', b', c')$ of non-negative integers. Once again, the spectra of ϕ^α and $\phi^{\alpha'}$ are not disjoint. $\square$

Proof of Theorem 17. If the condition holds, C_ϕ is an isometry by Lemma 18.

Conversely, suppose that C_ϕ is an isometry. Then, by Lemma 18, the spectra of ϕ^α and $\phi^{\alpha'}$ are disjoint if $\alpha \neq \alpha'$, and by Lemma 19 each ϕ_i is a monomial, necessarily of the form (27) by *(a)* of Lemma 18. Finally, if we denote by A the square matrix (a_{ij}), by $A^* = (a_{ji})$ its adjoint matrix, and if we let A, A^* act on $\mathbb{Z}^k$ by the formulas:

$$A(\alpha) = \beta\,, \qquad A^*(\alpha) = \gamma, \tag{29}$$

with $\beta_i = \sum_{j=1}^k a_{ij}\alpha_j$ and $\gamma_j = \sum_{i=1}^k a_{ij}\alpha_i$, we see that:

$$C_\phi(z^\alpha) = \phi^\alpha = \epsilon^\alpha z^{A^*(\alpha)}. \tag{30}$$

In fact,

$$C_\phi(z^\alpha) = \prod_i \phi_i^{\alpha_i} = \prod_i \epsilon_i^{\alpha_i}\Big(\prod_j z_j^{a_{ij}}\Big)^{\alpha_i} = \epsilon^\alpha \prod_j z_j^{\gamma_j}.$$

Now, by Lemma 18, the ϕ^α's have disjoint spectra, so that the $A^*(\alpha)$'s are distinct, implying $\det A \neq 0$. $\square$

If we now turn to the case of $A^+(\mathbb{T}^\infty)$, Lemma 18 clearly still holds, but Lemma 19 no longer holds : for example, if $I_1, \ldots, I_n, \ldots$ are disjoint subsets of $\mathbb{N}$, c_{ij} positive numbers such that $\sum_{j\in I_i} c_{ij} = 1$, $i = 1, 2, \ldots$ and if the map $\tilde\phi$ is defined by:

$$\tilde\phi = (\phi_i)_i, \quad \text{where } \phi_i(z) = \sum_{j\in I_i} c_{ij} z_j, \tag{31}$$

then $C_{\tilde\phi}$ is an isometry by Lemma 18 and yet no ϕ_i is a monomial if each I_i has more than one element. We have however a weaker result:

Theorem 20 *Let $\phi\colon \overline{\mathbb{D}}^\infty \to \overline{\mathbb{D}}^\infty$ be a map inducing a composition operator $C_\phi\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$, and such that moreover $\phi(\mathbb{T}^\infty) \subseteq \mathbb{T}^\infty$. Then:*

(a) There exists a matrix $A = (a_{ij})_{i,j\geqslant 1}$, with $a_{ij} \in \mathbb{N}_0$ and $\sum_j a_{ij} < \infty$ for each i, and complex signs ϵ_i such that $\phi = (\phi_i)_i$ and

$$\phi_i(z) = \epsilon_i \prod_{j=1}^\infty z_j^{a_{ij}}, \quad i = 1, 2, \ldots \tag{32}$$

(b) C_ϕ is an isometry if and only if $A^* = (a_{ji})$, acting on $\mathbb{Z}^{(\infty)}$ as in (29), is injective.

Proof. *(a)* If we apply Lemma 10 to the (connected) group $\Gamma = \mathbb{T}^\infty$ and its dual $G = \mathbb{Z}^{(\infty)}$, we see that for each $i \in \mathbb{N}$ there exists $L_i = (a_{i1}, a_{i2}, \ldots) \in \mathbb{Z}^{(\infty)}$, necessarily in $\mathbb{N}_0^{(\infty)}$, and a complex sign ϵ_i such that, for each $z \in \mathbb{T}^\infty$, we have:

$$\phi_i(z) = \epsilon_i < L_i, z > = \epsilon_i \prod_j z_j^{a_{ij}}$$

(note that, for $n \in \mathbb{N}$, setting $C = \|C_\phi\|$, we have

$$\|\phi_i^n\|_{A^+(\mathbb{T}^\infty)} = \|C_\phi(z_i^n)\|_{A^+(\mathbb{T}^\infty)} \leqslant C,$$

and also, since $|\phi_i(e^{it})| = 1$:

$$\|\phi_i^{-n}\|_{A^+(\mathbb{T}^\infty)} = \|\overline{\phi_i^n}\|_{A^+(\mathbb{T}^\infty)} = \|\phi_i^n\|_{A^+(\mathbb{T}^\infty)} \leqslant C).$$

This proves (32).

(b) We know from (30) (which clearly still holds for $k = \infty$) that $\phi^\alpha = \epsilon^\alpha z^{A^*(\alpha)}$, and we know from Lemma 18 that C_ϕ is an isometry if and only if the spectra of the ϕ^α's are disjoint. This gives the result. □

We shall prove here the announced converse of part *(a)* of Theorem 14.

Theorem 21 *Let $\phi = (\phi_j)_j \colon \boldsymbol{B} \to \boldsymbol{B}$ be an analytic function which induces a composition operator C_ϕ on $A^+(\mathbb{T}^\infty)$. If C_ϕ is an isometric automorphism of $A^+(\mathbb{T}^\infty)$, then $\phi(z) = (\epsilon_j z_{\sigma(j)})_j$, for some permutation σ of $\mathbb{N}$ and some some sequence $(\epsilon_j)_{j\geqslant 1}$ of complex signs.*

Proof. It suffices to look at the proof of Theorem 14, *(b)*: as in that proof, and with the same notation, it suffices to show that $\psi(\mathbf{B}) \subseteq \mathbf{B}$; but if it is not the case, it follows from (23), since the set J is infinite, that there exist at least two distinct integers $j_1, j_2 \in J$ such that the spectra of ϕ_{j_1} and ϕ_{j_2} are not disjoint. By Lemma 18, this contradicts the isometric nature of C_ϕ. □

Remark. It is easy to see that the composition operator $C_{\tilde\phi}$ on $A^+(\mathbb{T}^\infty)$ given by (31) does not correspond in general to a $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$.

For example, if

$$\phi_i(z) = \frac{z_{2i-1} + z_{2i}}{2}, \quad i = 1, 2, \ldots, \tag{33}$$

the equation $\tilde\phi(z^{[s]}) = z^{[\phi(s)]}$ would give:

$$\frac{p_{2i-1}^{-s} + p_{2i}^{-s}}{2} = p_i^{-\phi(s)}, i = 1, 2, \ldots;$$

taking equivalents of both members as $s \to \infty$ would give that

$$\frac{\phi(s)}{s} \underset{s\to+\infty}{\longrightarrow} \frac{\log p_{2i-1}}{\log p_i},$$

and it is impossible to have that, even for one i, since $\frac{\phi(s)}{s} \to c_0 \in \mathbb{N}_0$!

On the other hand, the additional assumption made in Theorem 20 does not allow to use the Bohr's transfer operator Δ to characterize the isometric composition operators on $\mathcal{A}^+$. Nevertheless, we have:

Theorem 22 *Let $\phi\colon \mathbb{C}_0 \to \mathbb{C}_0$ inducing a composition operator $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$. Then C_ϕ is an isometry if and only if $\phi(s) = c_0 s + i\tau$, with $c_0 \in \mathbb{N}$ and $\tau \in \mathbb{R}$.*

Proof. One direction is trivial. For the other, let us introduce the following notation: if $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{A}^+$, denote by $Sp\, f$ (the spectrum of f) the set of indices k such that $a_k \neq 0$. Now, the technique of the proof of Lemma 18 clearly works to show that:

$$\text{If } m \text{ and } n \text{ are distinct integers, the spectra of } m^{-\phi} \text{ and } n^{-\phi} \text{ are disjoint} \tag{34}$$

This automatically implies $c_0 \neq 0$, since, otherwise, the integer 1 would belong to the spectra of all the $n^{-\phi}$'s. Suppose now that ϕ is not of the form $c_0 s + c_1$, and write:

$$\phi(s) = c_0 s + c_1 + \omega(s),$$

with:

$$\omega(s) = c_r r^{-s} + c_{r+1}(r+1)^{-s} + \cdots, \quad r \geqslant 2, \quad c_r \neq 0.$$

Then:

$$\begin{aligned}
n^{-\phi(s)} &= (n^{c_0})^{-s} n^{-c_1} \exp\big(-\omega(s)\log n\big) \\
&= (n^{c_0})^{-s} n^{-c_1} \Big[1 + \sum_{k=1}^{\infty} \frac{(-\log n)^k}{k!} \big(\omega(s)\big)^k\Big] \\
&= (n^{c_0})^{-s} n^{-c_1} \Big[1 + \cdots + \sum_{k=1}^{\infty} \frac{(-\log n)^k}{k!} (c_r r^{-s} + \cdots)^k\Big].
\end{aligned}$$

For $\mathcal{R}e\, s$ large enough, all the series involved will be absolutely convergent; therefore the Dirichlet series of $n^{-\phi}$ will be obtained by expanding $(c_r r^{-s} + \cdots)^k$ and grouping terms. In particular, the coefficient λ_n of $n^{c_0} r^{c_0}$ in $n^{-\phi}$ can be obtained only by expanding $(c_r r^{-s} + \cdots)^k$ for $k = 1, \ldots, c_0$, so that $\lambda_n = P(\log n)$, where P is a non-zero polynomial. This implies that, for large n, $\lambda_n \neq 0$, and $(nr)^{c_0} \in Sp\, n^{-\phi}$. Moreover, it is clear that $l^{c_0} \in Sp\, l^{-\phi}$ for every positive integer l. Hence $(nr)^{c_0} \in Sp\, n^{-\phi} \cap Sp\,(nr)^{-\phi}$ for large n, which contradicts (34).

Therefore $\phi(s) = c_0 s + c_1$, and c_1 clearly has to be purely imaginary if C_ϕ is an isometry. $\square$

5 Concluding remarks and questions

Proposition 5 does not answer, in general, the natural question : if C_ϕ maps $\mathcal{A}^+$ into $\mathcal{A}^+$, is it true that $\phi(s) = c_0 s + \sum_1^{\infty} c_n n^{-s}$, with $\sum_1^{\infty} |c_n| < \infty$?

Proposition 7 does not apply to the case of complex coefficients c_r, c_{r^2}. Here, recent estimates due to Rusev [23] might help.

The estimate $\|\phi^n\|_{A^+} \geqslant \delta\sqrt{n}$ of Lemma 9 is best possible. In fact (see [14, p. 76]) it is fairly easy to see that $\|\phi^n\|_{A^+} \leqslant C\sqrt{n}$ if $\phi = e^{ig}$ and g is $\mathcal{C}^\infty$ (say), and a similar computation in dimension k (*i.e.* if we work with $A^+(\mathbb{T}^k)$) easily gives the estimate $\|\phi^n\|_{A^+(\mathbb{T}^k)} \leqslant C_k n^{k/2}$ if $\phi = e^{ig}$ and g is $\mathcal{C}^\infty$. It would be interesting to know whether the converse holds, *i.e.* if we have the following quantitative version of Lemma 9 : if $\phi = e^{ig}$, where g is a $\mathcal{C}^\infty$, non-affine, real function, then $\|\phi^n\|_{A^+} \geqslant \delta n^{1/2}$?

In the proof of Theorem 15, we used the fact that an analytic, almost-periodic, function defined on a vertical half-plane is never injective, to show that $c_0 > 0$, and therefore that the assumption *(b)* in Theorem 14 naturally holds. This raises two questions:

a) Can an almost-periodic function defined only on a vertical line be injective, *i.e.* can an almost-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ be injective? (of course, if f is real-valued, this is impossible: if f is injective, it is monotonic and therefore non almost-periodic).

b) Can one, at the price of using a different Banach-Stone type Theorem, dispense with the condition $\phi_k(z) = z_k^{d_k} u_k(z)$, with $d_k \geqslant 1$ and $u_k(0) \neq 0$ of *(b)* in Theorem 14, *i.e.* is the converse of *(a)* in this Theorem always true?

In view of the examples (31) in Section 4, a complete description of the isometric composition operators $C_\phi\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$ seems out of reach.

We gave a proof of Theorem 22 which does not use Theorem 17. Using this theorem, we can give a variant of Theorem 22: fix an integer $k \geqslant 1$, and denote by $\mathcal{A}_k^+$ the subalgebra of $\mathcal{A}^+$ consisting of the functions $f(s) = \sum_{P^+(n)\leqslant k} a_n n^{-s}$, where $P^+(n)$ denotes the largest prime factor of n. Equivalently, $f \in \mathcal{A}_k^+$ if the Dirichlet expansion of f only involves the primes $p_1, \dots, p_k$. Define similarly the subspace $\mathcal{D}_k$ of $\mathcal{D}$. With those definitions, we can state:

Theorem 23 *Let $\phi(s) = c_0 s + \varphi(s)$, $\varphi \in \mathcal{D}_k$, induce a composition operator $C_\phi\colon \mathcal{A}_k^+ \to \mathcal{A}_k^+$. Then $C_\phi\colon \mathcal{A}_k^+ \to \mathcal{A}_k^+$ is an isometry if and only if $\phi(s) = c_0 s + i\tau$, with $c_0 \in \mathbb{N}$ and $\tau \in \mathbb{R}$.*

Proof. The sufficiency is trivial. For the necessity, define an isometry $\Delta\colon \mathcal{A}_k^+ \to \mathcal{A}^+(\mathbb{T}^k)$ by:

$$\Delta\Big(\sum_{n=1}^\infty a_n n^{-s}\Big) = \sum_{n=1}^\infty a_n z_1^{\alpha_1} \dots z_k^{\alpha_k},$$

where $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the decomposition of n in prime factors. Set $z^{[s]} = (p_1^{-s}, \dots, p_k^{-s}) \in \mathbb{D}^k$ and check that $\Delta C_\phi \Delta^{-1} = T$ is a composition operator $C_{\tilde\phi}\colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$, isometric if C_ϕ is isometric, and such that:

$$\tilde\phi(z^{[s]}) = z^{[\phi(s)]}. \tag{35}$$

We now use Theorem 17 to conclude that $\tilde\phi = (\phi_1, \dots, \phi_k)$, with $\phi_1(z) = \epsilon_1 z_1^{a_{11}} \dots z_k^{a_{1k}}$, and where $a_{11}, \dots, a_{1k}$ are non-negative integers. Exactly as in the Proof of Theorem 11, we then conclude that $\phi(s) = c_0 s + i\tau$. □

In the next Theorem, we shall see that there are few composition operators whose symbols preserve the boundary $i\mathbb{R}$.

Theorem 24 *Let $\phi\colon \mathbb{C}_0 \to \mathbb{C}_0$ inducing a composition operator $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$, and such that moreover ϕ has a continuous extension to $\overline{\mathbb{C}}_0$, preserving the boundary of $\mathbb{C}_0$, i.e. $\phi(i\mathbb{R}) \subseteq i\mathbb{R}$. Then $\phi(s) = c_0 s + i\tau$, where $c_0 \in \mathbb{N}_0$ and $\tau \in \mathbb{R}$.*

Proof. Let $\tilde\phi$ be associated with ϕ as in Theorem 11. By continuity, the equation $\tilde\phi\big(z^{[s]}\big) = z^{[\phi(s)]}$, $s \in \mathbb{C}_0$, still holds for $s = it$, $t \in \mathbb{R}$, to give $\tilde\phi\big((p_j^{-it})_j\big) = \big(p_j^{-\phi(it)}\big)_j$, and so $\tilde\phi(\mathbb{T}^\infty) \subseteq \mathbb{T}^\infty$ since, by the Kronecker Approximation Theorem and the definition of the product topology on $\mathbb{T}^\infty$, the points $(p_j^{-it})_j$, $t \in \mathbb{R}$, are dense in $\mathbb{T}^\infty$. Now, by Theorem 20, we have in particular $\tilde\phi = (\phi_i)_i$, with

$$\phi_1(z) = \epsilon_1 z_1^{a_{11}} \dots z_k^{a_{1k}},$$

for some complex sign ϵ_1 and some integer k. In particular, the equation $\tilde\phi(z^{[s]}) = z^{[\phi(s)]}$ implies that:

$$\epsilon_1 (p_1^{-s})^{a_{11}} \dots (p_k^{-s})^{a_{1k}} = p_1^{-\phi(s)}, \quad s \in \mathbb{C}_0.$$

Passing to the moduli gives $\mathcal{R}e\,\phi(s) = c\,\mathcal{R}e\,s$, with $c = \sum_{j=1}^k a_{1j} \frac{\log p_j}{\log p_1}$.

Theorefore, $\phi(s) - cs = i\tau$, $\tau \in \mathbb{R}$, and we know that $c = c_0$ is necessarily an integer. $\square$

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